universal (relative) unipotent objects and ~~some~~ one application~~s~~

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Theorem of Andreatta-Iovita-Kim

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Let $k/\mathbb{Q}_p$ finite. Let X be a curve over k with genus ≥ 2 that has semi-stable reduction. Then X has good reduction if and only if

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key insights:

- $\pi_1^{\text{ét,uni}}(X_{\bar{k}}, \bar{x}) = \mathrm{Spec}(A_\infty^{\text{ét},\vee})$, with

$$A_\infty^{\text{ét},\vee} = \lim \mathbb{L}_{n,x}^\vee,$$

  with $\mathbb{L}_{n,x}$ semi-stable, "universal unipotent" Galois-representations.
- can understand relevant monodromy operator on $\mathrm{D}_{\mathrm{st}}(\mathbb{L}_{n,x})$ after base change along an embedding $k \hookrightarrow \mathbb{C}$.

universal unipotent objects

$\mathcal{T}$ – finitely generated unipotent Tannakian category

Lemma (Andreatta-Iovita-Kim)

A pointed pro-system (V_n, v_n) *represents* ω *iff*

(i) $V_0 = \mathbf{1}$, $e_n = 1$,

(ii) $f_n : V_{n+1} \to V_n$ *is surjective with constant kernel* T_n

(iii) *the boundary map*

$$\mathrm{Hom}(T_n, \mathbf{1}) \longrightarrow \mathrm{Ext}^1(V_n, \mathbf{1})$$

is an isomorphism

relative unipotent objects, following Betts, Lazda

- *relative Tannakian category*: $\pi^* : \mathcal{T}_S \to \mathcal{T}_X$ fully faithful exact $\otimes$-functor between Tannakian categories, such that the essential image is closed under subobjects
- *relative fiber functor*: $x^* : \mathcal{T}_X \to \mathcal{T}_S$ exact tensor functor and isomorphism $x^*\pi^* \xrightarrow{\sim} \mathrm{id}$

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$$\mathrm{Hom}_{\mathcal{T}_S}(x^*(-), \mathbf{1}_S) : \text{ind-}\mathcal{T}_X \to \mathrm{Set}$$

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- 🥳: these technical problems can be overcome by a careful analysis of Betts’ work

Application over $\mathbb{Q}_p$

Proposition (w.)

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(i) *The relative universal $\mathbb{L}_n \in \mathrm{Loc}_{\mathbb{Q}_p}(X)$ are de Rham*

(ii) *For $\mathcal{E}_n$ universal relative filtered vector bundles on X with integrable connection:*

$$\mathrm{D}_{\mathrm{dR}}(\mathbb{L}_n) \xrightarrow{\sim} \mathcal{E}_n$$

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- uses Betts' earlier result for $S = \mathrm{pt}$ as input (maybe there are more geometric ways to prove this using methods of Wojtkowiak)
- again amenable to base change along $k \hookrightarrow \mathbb{C}$.