TALK 4: HODGE–TATE SECTIONS

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Goal: Introduce the notion of Hodge–Tate-ness for sections $s\colon \mathrm{Gal}_K \to \pi_1(X,x)$ of the fundamental exact sequence of étale fundamental groups

$$1 \to \pi_1(X_{\overline{K}}, x) \to \pi_1(X,x) \to \mathrm{Gal}_K \to 1.$$

Let

- K be the fraction field of a complete DVR, mixed characteristic $(0,p)$,
- $\mathbb{C}_K = \widehat{\overline{K}}$,
- X a smooth projective geometrically connected (hyperbolic) curve over K,
- x a geometric point of X,
- $G := \pi_1^{\mathbb{Q}_p}(X_{\overline{K}}, x)$ the $\mathbb{Q}_p$-prounipotent fundamental group of $X_{\overline{K}}$,
- $\mathfrak{g}$ the Lie algebra of G (a pronilpotent Lie algebra over $\mathbb{Q}_p$),
- $Z^0(G) = G$, $Z^n(G) = [G, Z^{n-1}(G)]$, $Z^0(\mathfrak{g}) = \mathfrak{g}$, $Z^n(\mathfrak{g}) = [\mathfrak{g}, Z^{n-1}(\mathfrak{g})]$,
- $\mathrm{gr}^n_Z(G) = Z^n(G)/Z^{n+1}(G)$ (so that $\mathrm{gr}^0_Z(G) = G^{\mathrm{ab}}$), sitting in the exact sequence

$$1 \to \mathrm{gr}^n_Z(G) \to G/Z^{n+1}(G) \to G/Z^n(G) \to 1. \tag{0.1}$$

In the last talk, Leonie introduced two (equivalent) descriptions of the prounipotent group G over $\mathbb{Q}_p$:

(1) as the Tannaka group of the category of unipotent $\mathbb{Q}_p$-local systems on $X_{\overline{K},\text{ét}}$ equipped with the fibre functor "fibre over x".
(2) as the $\mathbb{Q}_p$-Malcev completion of the étale fundamental group $\pi_1(X_{\overline{K}}, x)$: There is a continuous group homomorphism $\varphi\colon \pi_1(X_{\overline{K}}, x) \to G(\mathbb{Q}_p)$ that is universal for continuous group homomorphisms of $\pi_1(X_{\overline{K}}, x)$ into the $\mathbb{Q}_p$-points of a (pro-)unipotent group over $\mathbb{Q}_p$.

1. Sections and Galois action on G

We use the functoriality of the following functors

$$\begin{aligned}
\text{(Pointed "nice" Schemes)} &\longrightarrow \text{(Profinite Groups)}, & (Y,y) &\longmapsto \pi_1(Y,y),\\
\text{(Profinite Groups)}^{\mathrm{op}} &\longrightarrow \text{(unipotent Tannaka cat's}/\mathbb{Q}_p), & \pi &\longmapsto (\mathrm{Rep}^{\mathrm{un}}_{\mathbb{Q}_p}(\pi), ?),\\
\text{(unipotent Tannaka cat's}/\mathbb{Q}_p)^{\mathrm{op}} &\longrightarrow \text{(prounipotent groups}/\mathbb{Q}_p), & (\mathcal{T},\omega) &\longmapsto \pi_1(\mathcal{T},\omega),\\
\text{(prounipotent groups}/\mathbb{Q}_p) &\longrightarrow \text{(pronilpotent Lie algebras}/\mathbb{Q}_p), & H &\longmapsto \mathrm{Lie}(H).
\end{aligned}$$

Construction 1.1. Let $s\colon \mathrm{Gal}_K \to \pi_1(X,x)$ be a section. Then every $\sigma \in \mathrm{Gal}_K$ induces the automorphism "conjugation by $s(\sigma)$" on $\pi_1(X_{\overline{K}}, x)$. Using the functors above, this in turn induces an automorphism of G (of prounipotent groups) and of $\mathfrak{g}$ (of Lie algebras). By compatibility with composition (i.e. functoriality), this defines an action of Gal_K on G and on $\mathfrak{g}$ *that depends on s.* If we like to stress the dependency on s, we denote this action by $\sigma \cdot_s -$.

We think of $\mathfrak{g}$ as a fixed (pro-finite dimensional) $\mathbb{Q}_p$-vector space on which, for every choice of section s, the group Gal_K acts. We aim to understand what kind of representations may arise from an arbitrary section or, more specifically, from a section induced from a rational point.

Remark 1.2. Given two sections s, s', for every $\sigma \in \mathrm{Gal}_K$ the elements $s(\sigma)$ and $s'(\sigma)$ differ by an element $a_\sigma \in \pi_1(X_{\overline{K}}, x)$, i.e.

$$s'(\sigma) = a_\sigma s(\sigma).$$

The $(a_\sigma)_\sigma$ form a 1-cocycle of Gal_K with values in $\pi_1(X_{\overline{K}}, x)$ (equipped with the action coming from s), i.e.

$$a_{\sigma\tau} = a_\sigma(\sigma \cdot_s a_\tau), \quad \sigma, \tau \in \mathrm{Gal}_K .$$

Example 1.3. Conjugation by $a \in \pi_1(X_{\overline{K}}, x)$ on $\pi_1(X_{\overline{K}}, x)$ induces conjugation by $\varphi(a) \in G(\mathbb{Q}_p)$ on G and hence $\mathrm{Ad}(\varphi(a))$ on $\mathfrak{g}$, where $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ denotes the adjoint action. By the previous remark, we have $(s'(\sigma) - s'(\sigma)^{-1}) = (a_\sigma - a_\sigma^{-1}) \circ (s(\sigma) - s(\sigma)^{-1})$ as automorphisms of $\pi_1(X_{\overline{K}}, x)$, so by functoriality we have

$$\sigma \cdot_{s'} v = \mathrm{Ad}(\varphi(a_\sigma))(\sigma \cdot_s v), \quad \sigma \in \mathrm{Gal}_K, v \in \mathfrak{g}. \tag{1.1}$$

Remark 1.4. The subgroups $Z^n(G)$ are characteristic, hence they are Gal_K-stable and we get (for every section s) an induced action of Gal_K on $G/Z^n(G)$ and on $\mathrm{gr}^n_Z(G)$. As $G/Z^n(G)$ is the Tannaka fundamental group of the category of unipotent representations of $\pi_1(X_{\overline{K}}, x)$ (or of G) of unipotency class $\leq n$, this corresponds to the action of σ on the category of unipotent representations respecting the unipotency class. Similarly for $\mathfrak{g}/Z^n(\mathfrak{g})$ and $\mathrm{gr}^n_Z(\mathfrak{g})$.

Remark 1.5. If u is a nilpotent endomorphism of a finite dimensional $\mathbb{Q}_p$-vector space V, then

$$\exp(u) := 1 + u + \frac{u^2}{2!} + \dots$$

is a finite sum and hence a well-defined unipotent automorphism of V. Doing this functorially gives a well-defined *exponential map*

$$\exp\colon \mathfrak{g} \longrightarrow G$$

for any unipotent group G over $\mathbb{Q}_p$. This map is an isomorphism of schemes (where $\mathfrak{g}$ denotes the functor $R \mapsto \mathfrak{g} \otimes_{\mathbb{Q}_p} R$) and can be upgraded to an isomorphism of algebraic groups if we equip $\mathfrak{g}$ with the multiplication given by the Baker-Campbell-Hausdorff formula: a certain (finite) series involving iterated Lie brackets only. See [Milne, Algebraic Groups, Ch. 15, i.p. Prop. 15.35] for details. In this way, we even get a category equivalence

$$(\text{Unipotent algebraic groups over } \mathbb{Q}_p) \longrightarrow (\text{Nilpotent Lie algebras over } \mathbb{Q}_p).$$

Important for us is the following relationship between the adjoint representation and the exponential:

$$\mathrm{Ad}(\exp(v)) = \exp(\mathrm{ad}(v)), \quad v \in \mathfrak{g}, \tag{1.2}$$

(equality of automorphisms of $\mathfrak{g}$), where $\mathrm{ad}(v) = [v, -]$. See [Hall, Lie Groups, Lie Algebras, and Representations, Prop. 3.35]. We silently assume that this all works also for the pro-versions, and moreover we stop distinguishing between G (resp. $Z^n(G)$, $\mathrm{gr}^n_Z(G)$) and $\mathfrak{g}$ (resp. $Z^n(\mathfrak{g})$, $\mathrm{gr}^n_Z(\mathfrak{g})$).

Lemma 1.6. *The* Gal_K*-action on* $\mathrm{gr}^n_Z(\mathfrak{g})$ *is independent of the section* s.

Proof. There was a discussion during class about two possible proofs. Since we need both points of view later on anyway, let us present both proofs, one direct, one using functoriality.

(1) (direct) By (1.1) and (1.2), we see that

$$\sigma \cdot_{s'} v = \sigma \cdot_s v + (\text{terms involving iterated Lie brackets between } b_\sigma \text{ and } \sigma \cdot_s v),$$

where $b_\sigma \in \mathfrak{g}$ is any element with $\exp(b_\sigma) = \varphi(a_\sigma)$. If v lies in $Z^n(\mathfrak{g})$, so do $\sigma \cdot_{s'} v$ and $\sigma \cdot_s v$. Hence all extra terms in the above equality lie in $Z^{n+1}(\mathfrak{g})$, thus the action on $\mathrm{gr}^n_Z(\mathfrak{g})$ is independent of s.

(2) (via functoriality) The Abel-Jacobi map from X into its Jacobian J induces, on étale fundamental groups, a Gal_K-equivariant isomorphism $\pi_1(X_{\overline{K}}, x)^{\mathrm{ab}} \cong \pi_1(J_{\overline{K}}, x)$. Since $\pi_1(J_{\overline{K}}, x)$ is abelian, the Gal_K-action on it is independent of the section s. On the $\mathbb{Q}_p$-prounipotent level, we get a Gal_K-equivariant isomorphism

$$\mathrm{gr}^0_Z(G) = G/[G,G] \cong \pi_1^{\mathbb{Q}_p}(J_{\overline{K}}, x),$$

proving the case $n = 0$. For $n \geq 1$, use that (from the description of the geometric étale fundamental group as the profinite completion of a surface group) one has a Gal_K-equivariant surjection

$$\bigoplus_{\text{orderings } \Delta \text{ of Lie brackets}} T^{\otimes n}\, \mathrm{gr}^0_Z(\mathfrak{g}) \longrightarrow \mathrm{gr}^n_Z(\mathfrak{g}), \quad v_1 \otimes \cdots \otimes v_n \longmapsto [\ldots[[v_1, v_2], v_3]\ldots] \tag{1.3}$$

(where the Lie brackets are placed depending on Δ) showing the desired independence of s. □

Remark 1.7. While the Gal_K-action on the graded pieces is independent of s, the Gal_K-action on $G/Z^n(G)$ may depend on s! Even putting $n = 1$ in (0.1), where both the left hand and the right hand term carry Gal_K-actions independent of s, the action on the middle term may still depend on s.

Example 1.8. We can (at least) make $\mathrm{gr}^0_Z(\mathfrak{g})$ more explicit. We have a Gal_K-equivariant identification

$$\mathrm{gr}^0_Z(\mathfrak{g}) = V_p(J),$$

the p-adic Tate module of J tensored with $\mathbb{Q}_p$. Why? In the second proof above, we showed $\mathrm{gr}^0_Z(\mathfrak{g}) \cong \pi_1^{\mathbb{Q}_p}(J_{\overline{K}}, x)$. The latter is the Malcev completion of $\pi_1(J_{\overline{K}}, x)$, the full Tate module $\widehat{T}J = (\varprojlim_n J(\overline{K})[n])$ of J. It is now straightforward to see that every unipotent representation of $\widehat{T}J$ over $\mathbb{Q}_p$ factors through $V_p(J)$, showing the desired result.

Conclusion 1.9. So far we have seen that the map

$$(\text{sections}) \to (\mathrm{Gal}_K\text{-representations on } \mathfrak{g})$$

lands inside those representations that respect the descending central series filtration $Z^\bullet(\mathfrak{g})$ and, for every n, induce a specified representation on the graded piece $\mathrm{gr}^n_Z(\mathfrak{g})$.

2. Hodge–Tate sections

Remark 2.1. All $\mathrm{gr}^n_Z(\mathfrak{g})$ are Hodge–Tate representations of Gal_K of Hodge–Tate weights between $-n$ and 0. Why? For $n = 0$, this follows from Example 1.8 and the Hodge–Tate decomposition of the Tate module of an abelian variety: We have a Gal_K-equivariant isomorphism of $\mathbb{C}_K$-semilinear representations

$$V_p(J) \otimes_{\mathbb{Q}_p} \mathbb{C}_K = \left(H^0(X, \Omega^1_{X/K})^\vee \otimes_K \mathbb{C}_K(1)\right) \oplus \left(H^1(X, \mathcal{O}_X) \otimes_K \mathbb{C}_K\right), \tag{2.1}$$

showing that $V_p(J)$ is Hodge–Tate of weights -1 and 0 (each with multiplicity g). For $n \geq 1$, the claim follows from (1.3).

However, this does not imply that the action on $\mathfrak{g}/Z^n(\mathfrak{g})$ is Hodge–Tate, as Hodge–Tate-ness is not preserved under extensions.

For now, we will use the following result as a black box:

Theorem 2.2. *Let $s = s_x$ for a point $x \in X(K)$ (lying under the fixed geometric point x). Then the induced action of Gal_K on $\mathfrak{g}/Z^n(\mathfrak{g})$ is Hodge–Tate, i.e. there is a Gal_K-equivariant decomposition*

$$(\mathfrak{g}/Z^n(\mathfrak{g})) \otimes_{\mathbb{Q}_p} \mathbb{C}_K = \bigoplus_{i=0}^{n} \mathbb{C}_K(i)^{d_n(i)}$$

for some $d_n(i) \in \mathbb{N}_0$.

We fix such an $x \in X(K)$ and its corresponding section s_x for the rest of this section[1].

Definition 2.3. We define a decreasing filtration $E^\bullet$ on $(\mathfrak{g}/Z^n(\mathfrak{g})) \otimes_{\mathbb{Q}_p} \mathbb{C}_K$ by

$$E^m(\mathfrak{g}/Z^n(\mathfrak{g})) = \bigoplus_{i=m}^{n} \mathbb{C}_K(i)^{d_n(i)},$$

i.e. E^m is the part of Hodge–Tate weight $\leq -m$ under the Gal_K-action induced by s_x.

Remark 2.4. The filtration $E^\bullet$ is finite, separated and exhaustive, with $E^{n+1} = 0$ and $E^0 = (\mathfrak{g}/Z^n(\mathfrak{g})) \otimes_{\mathbb{Q}_p} \mathbb{C}_K$. Also the E^m are Lie ideals, as Gal_K acts (via s) by Lie algebra homomorphisms on $\mathfrak{g}$ and hence also on $(\mathfrak{g}/Z^n(\mathfrak{g})) \otimes_{\mathbb{Q}_p} \mathbb{C}_K$: If $e \in E^m$ (wlog of HT weight $-j \leq -m$) and $g \in (\mathfrak{g}/Z^n(\mathfrak{g})) \otimes_{\mathbb{Q}_p} \mathbb{C}_K$ (wlog of HT weight $-i \leq 0$), then for any $\sigma \in \mathrm{Gal}_K$ we have $\sigma \cdot_s [g, e] = [\sigma \cdot_s g, \sigma \cdot_s e] = \chi(\sigma)^{i+j}[g, e]$, i.e. $[g, e]$ has HT weight $-(i + j) \leq -m$ and thus lies in E^m again.

We want to show an "independence of s"-result for the filtration $E^\bullet$:

Lemma 2.5. *For any section* $s' \colon \mathrm{Gal}_K \to \pi_1(X, x)$*, the subspace* E^m *of* $(\mathfrak{g}/Z^n(\mathfrak{g})) \otimes_{\mathbb{Q}_p} \mathbb{C}_K$ *is* Gal_K*-stable for the action induced by* s'*.*

Proof. We need to show that if $e \in E^m$ (wlog of HT weight $-i \leq -m$) and $\sigma \in \mathrm{Gal}_K$, then $\sigma \cdot_{s'} e$ lies in E^m. Using (1.1) and (1.2) we see that

$$\begin{aligned}\sigma \cdot_{s'} e &= \sigma \cdot_s e + (\text{terms involving iterated Lie brackets between } b_\sigma \text{ and } \sigma \cdot_s e) \\ &= \chi(\sigma)^i (e + (\text{terms involving iterated Lie brackets between } b_\sigma \text{ and } e)),\end{aligned}$$

where $b_\sigma \in \mathfrak{g}$ is any element with $\exp(b_\sigma) = \varphi(a_\sigma)$. As E^m is a Lie ideal, all extra terms in the above equality lie in E^m, thus $\sigma \cdot_{s'} e$ lies in E^m as well. ☐

Remark 2.6. At this point, we have *not yet* shown that the definition of the space E^m (as the part of Hodge–Tate weight $\leq -m$ under the Gal_K-action via s_x) is independent of the chosen section s_x. The previous lemma only shows that the space E^m is Gal_K-stable for the action induced by any section s', not that E^m is also the space of Hodge–Tate weight $\leq -m$ for the Gal_K-action induced by s'!

We remedy this confusion by proving the following claim: If s' is any section such that

$$\mathfrak{g}/Z^n(\mathfrak{g}) \text{ is Hodge–Tate for all } n \tag{2.2}$$

(e.g. $s' = s_y$ for some $y \in X(K)$), then the corresponding subspace $E'^{,m}$ of Hodge–Tate weight at most $-m$ w.r.t. s' is equal to E^m. This also shows that the action of Gal_K on $(\mathfrak{g}/Z^n(\mathfrak{g})) \otimes_{\mathbb{Q}_p} \mathbb{C}_K$ is independent of the chosen section s' satisfying (2.2).

Proof of the claim. We proceed by induction on n, and for fixed n, by decreasing induction on m. The case $n = 1$ is OK since the Gal_K-action on $\mathfrak{g}/Z^1(\mathfrak{g}) = \mathrm{gr}^0_Z(\mathfrak{g})$ is independent of s'. We only prove the case $n = 2$, which indicates how the general proof works, but avoids too much index fighting. We use (0.1) (tensored with $\mathbb{C}_K$) to get a short exact sequence of $\mathbb{C}_K$-vector spaces of the form

$$0 \to A \to B \to C \to 0$$

that are also short exact sequences of $\mathbb{C}_K[\mathrm{Gal}_K]$-modules (with semilinear actions) for two different Gal_K-actions on B, being induced by s and s' respectively, but for the same actions on A and C. Here $A = \mathrm{gr}^1_Z(\mathfrak{g}) \otimes_{\mathbb{Q}_p} \mathbb{C}_K$, $B = (\mathfrak{g}/Z^2(\mathfrak{g})) \otimes_{\mathbb{Q}_p} \mathbb{C}_K$ and $C = \mathrm{gr}^0_Z(\mathfrak{g}) \otimes_{\mathbb{Q}_p} \mathbb{C}_K$. The three

[1] If X does not have a K-rational point, we may go to a finite extension because Hodge–Tate-ness (and everything we do in this section) is insensitive to finite field extensions.

representations carry filtrations $E^m(A)$, $E^m(C)$ (both independent of s'), $E^m(B)$ w.r.t. s and $E'^{,m}(B)$ w.r.t. s'. The picture is like this

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & & B & & \longrightarrow & C & \longrightarrow & 0 \\
& & \uparrow & & \nearrow & & \nwarrow & & \uparrow & & \\
& & E^1(A) & & E^1(B) & & E'^{,1}(B) & & E^1(C) & & \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
& & E^2(A) & & E^2(B) & & E'^{,2}(B) & & E^2(C)=0 & & \\
& & \uparrow & & \uparrow & & \uparrow & & & & \\
& & E^3(A)=0 & & E^3(B)=0 & & 0=E'^{,3}(B) & & & &
\end{array}$$

The horizontal arrows are Gal_K-equivariant, hence respect Hodge–Tate weights (as there are no non-trivial homomorphisms $\mathbb{C}_K(i) \to \mathbb{C}_K(j)$ unless $i = j$) no matter which section we choose for the action on B. This shows that $E^2(B) = E^2(A) = E'^{,2}(B)$. But then also $E^1(B) = E'^{,1}(B)$. This shows that the action on B is independent of the choice of s'. □

The filtrations $E^\bullet$ on $(\mathfrak{g}/Z^n(\mathfrak{g})) \otimes_{\mathbb{Q}_p} \mathbb{C}_K$ for varying n are also compatible, inducing a filtration $E^\bullet$ on $\mathfrak{g}$. We are now ready for the main definition of this talk:

Definition 2.7. We let $\mathfrak{h} := \mathrm{gr}^0_E(\mathfrak{g} \otimes_{\mathbb{Q}_p} \mathbb{C}_K)$. A section s' is called *Hodge–Tate* if $\mathfrak{h}$ equipped with the Gal_K-action via s' is Hodge–Tate of Hodge-Tate weight 0, i.e. if $\mathfrak{h}$ is generated as a $\mathbb{C}_K$-vector space by its Gal_K-invariants.

Example 2.8. (1) The fixed section $s = s_x$ is Hodge–Tate. Why? By Theorem 2.2, the Gal_K-representation $\mathfrak{g} \otimes_{\mathbb{Q}_p} \mathbb{C}_K$ via s is Hodge–Tate. By definition, $\mathfrak{h}$ is the part (more accurately, quotient) of $\mathfrak{g} \otimes_{\mathbb{Q}_p} \mathbb{C}_K$ of Hodge–Tate weight 0 under the Gal_K-action induced by s, so $\mathfrak{h}$ is Hodge–Tate of weight 0.

(2) Using (2.1), we have a Gal_K-equivariant isomorphism
$$\mathrm{gr}^0_E\left((\mathfrak{g}/Z^1(\mathfrak{g})) \otimes_{\mathbb{Q}_p} \mathbb{C}_K\right) = H^1(X, \mathcal{O}_X) \otimes_K \mathbb{C}_K,$$
so taking gr^0_E picks out half of the generators of $(\mathfrak{g}/Z^1(\mathfrak{g})) \otimes_{\mathbb{Q}_p} \mathbb{C}_K$. In fact, the Lie algebra $\mathfrak{h}$ is a free pro-nilpotent Lie algebra in g generators over $\mathbb{C}_K$.

Proposition 2.9. *If $s' = s_y$ for a K-rational point $y \in X(K)$, then s' is Hodge–Tate.*

Proof. By[2] Theorem 2.2, the Gal_K-action on $\mathfrak{g} \otimes_{\mathbb{Q}_p} \mathbb{C}_K$ induced by s' is Hodge–Tate. By Remark 2.6, this means that the Gal_K-action on $E'^{,m} = E^m$ induced from s' is the same as the one induced from s, so $\mathfrak{h}$ is Hodge–Tate by Example 2.8, part (1). □

Conclusion 2.10. Let us continue the point of view of Conclusion 1.9. The black-box Theorem 2.2, the notion of Hodge–Tate sections, and Proposition 2.9 reveal the finer structure of the map from Conclusion 1.9:

$$\begin{array}{ccc}
(\text{sections}) & \longrightarrow & (\mathrm{Gal}_K\text{-rep's on } \mathfrak{g} \text{ respecting } Z^\bullet(\mathfrak{g}) \text{ with fixed rep's on } \mathrm{gr}^n_Z(\mathfrak{g})) \\
\uparrow & & \uparrow \\
(\text{Hodge–Tate sections}) & \longrightarrow & (\mathrm{Gal}_K\text{-rep's on } \mathfrak{g} \text{ as above s.t. } \mathfrak{h} \text{ is Hodge–Tate of weight } 0) \\
\uparrow & & \uparrow \\
X(K) & \longrightarrow & (\mathrm{Gal}_K\text{-rep's on } \mathfrak{g} \text{ as above that satisfy } (2.2)).
\end{array}$$

It seems reasonable to call a representation on $\mathfrak{g}$ *pro-Hodge–Tate* if it satisfies (2.2).

[2] There is a tiny catch: In Theorem 2.2 we assume that the point x lies below the chosen geometric point x.

3. de Rham comparison for G

It remains to prove Theorem 2.2, which follows directly from the following "unipotent de-Rham comparison isomorphism" [Betts, Local Constancy of prounipotent Kummer maps, Theorem 1.4(1)].

Theorem 3.1. *Let $K/\mathbb{Q}_p$ be a finite extension*[3]*. Let $s = s_x$ be induced from a rational point $x \in X(K)$. Then there is a canonical Gal_K-equivariant isomorphism*

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} G \cong B_{\mathrm{dR}} \otimes_K \pi_1^{\mathrm{dR}}(X, x)$$

of B_{dR}-schemes, where Gal_K-acts on G via s and naturally on B_{dR}.

Here, $\pi_1^{\mathrm{dR}}(X, x)$ is the Tannaka fundamental group of the category $\mathrm{MIC}^{\mathrm{un}}(X, \mathcal{O}_X)$ of unipotent vector bundles with integrable connection on X w.r.t. the fibre functor ω_x.

Proof of Theorem 2.2. On the right hand side of the comparison isomorphism, Gal_K acts only on B_{dR}. Thus Theorem 3.1 shows that G is a pro-de Rham representation, hence is pro-Hodge–Tate[4], which is precisely the claim in Theorem 2.2. □

Let us give a sketch of the proof of Theorem 3.1, indicating where results from Leonie's talk play a role, but not diving into full details. For simplicity of exposition, we continue to work with smooth projective curves X, though in fact Betts proves the comparison isomorphism for an arbitrary smooth scheme X/K. Betts compares the respective Tannaka categories after analytification[5] with a third Tannaka category called the pro-unipotent Riemann–Hilbert category and denoted $\mathrm{MIC}^{\mathrm{un}}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$:

$$\begin{array}{ccc} & \mathrm{MIC}^{\mathrm{un}}(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}) & \\ \nearrow^{\mathcal{RH}} & & \nwarrow^{B_{\mathrm{dR}}\hat{\otimes}_K -} \\ (\text{unipotent } \mathbb{Q}_p\text{-local systems on } X^{\mathrm{an}}_{\mathbb{C}_K,\text{ét}}) & & \mathrm{MIC}^{\mathrm{un}}(X^{\mathrm{an}},\mathcal{O}_{X^{\mathrm{an}}}) \\ \sim\uparrow & & \uparrow\sim \\ (\text{unipotent } \mathbb{Q}_p\text{-local systems on } X_{\overline{K},\text{ét}}) & & \mathrm{MIC}^{\mathrm{un}}(X,\mathcal{O}_X)\end{array}$$

3.1. Structure of $\mathrm{MIC}^{\mathrm{un}}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$.

- $\mathrm{MIC}^{\mathrm{un}}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is a finitely generated unipotent Tannakian category with fibre functor ω_x^{dR}. Key ingredient here is the calculation of Ext^1 in this category in terms of H^1_{dR}, and the following comparison for the tensor unit $\mathcal{O}_{\mathfrak{X}}$:

$$B_{\mathrm{dR}} \otimes_K H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \xrightarrow{\sim} H^1_{\mathrm{dR}}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \tag{3.1}$$

  We denote the Tannaka fundamental group of $\mathrm{MIC}^{\mathrm{un}}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ by $\pi_1^{\mathrm{RH}}(\mathfrak{X}, x)$.
- Gal_K acts on B_{dR}, thus on $\mathcal{O}_{\mathfrak{X}}$, thus on $\mathrm{MIC}^{\mathrm{un}}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ and hence on $\pi_1^{\mathrm{RH}}(\mathfrak{X}, x)$.
- By equipping the pro-representing object $\mathcal{E}^{\mathrm{RH}}$ (see Leonie's talk) of ω_x^{dR} with a suitable filtration, one can equip $\pi_1^{\mathrm{RH}}(\mathfrak{X}, x)$ with the so-called Hodge filtration.

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[3] Betts makes this assumption in his article, though I am not sure it is necessary. But as written, there is a discrepancy with our assumption.

[4] Up to unravelling the "pro", this fits into the usual formalism of HT/dR/crystalline representations as "those with a comparison isomorphism", compare Morten's talk 5.

[5] The vertical category equivalences follow from results of Illusie and rigid GAGA, respectively.

3.2. Comparison on de Rham side.

Lemma 3.2. *There is a* Gal_K*-equivariant, strictly filtration-compatible isomorphism*

$$\pi_1^{\mathrm{RH}}(\mathfrak{X}, x) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_K \pi_1^{\mathrm{dR}}(X^{\mathrm{an}}, x).$$

Sketch of proof. Use the criterion of talk 3. For an isomorphism as claimed, we only need to establish an isomorphism on Ext^1's, which can be expressed in terms of H^1_{dR}. For the trivial bundle $\mathcal{O}_{X^{\mathrm{an}}}$, this is (3.1); for arbitrary objects, use induction by the unipotency class. Compatibility with Gal_K-action and Hodge filtrations are slightly more technical. □

3.3. Comparison on étale side.

Here, Betts goes via the proétale site: There are morphisms of sites

$$X^{\mathrm{an}}_{\mathbb{C}_K,\text{ét}} \xleftarrow{\overline{\nu}} X^{\mathrm{an}}_{\mathbb{C}_K,\text{proét}} \xrightarrow{\overline{\mu}} \mathfrak{X}.$$

He introduces another finitely generated unipotent Tannakian category of "unipotent $\hat{\mathbb{Q}}_p$-local systems" on $X^{\mathrm{an}}_{\mathbb{C}_K,\text{proét}}$ together with a *period sheaf* $\mathcal{O}\mathbb{B}^{\wedge}_{\mathrm{dR}}$ and defines the Riemann–Hilbert functor as

$$\mathcal{RH}(\overline{\mathbb{E}}) := \overline{\mu}_* \left(\mathcal{O}\mathbb{B}^{\wedge}_{\mathrm{dR}} \otimes_{\hat{\mathbb{Q}}} \widehat{\overline{\mathbb{E}}} \right).$$

This functor is constructed in such a way that there is a comparison isomorphism

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^i_{\text{ét}}(X^{\mathrm{an}}_{\mathbb{C}_K}, \overline{\mathbb{E}}) \xrightarrow{\sim} H^i_{\mathrm{dR}}(\mathfrak{X}, \mathcal{RH}(\overline{\mathbb{E}})).$$

After taking care of Gal_K-descent, the proof of the following Lemma is then similar to that of Lemma 3.2.

Lemma 3.3. *There is a* Gal_K*-equivariant, strictly filtration-compatible isomorphism*

$$\pi_1^{\mathrm{RH}}(\mathfrak{X}, x) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} G$$

Together, Lemma 3.2 and Lemma 3.3 prove Theorem 3.1.